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Short Communication

Application of the harmonic balance method to a nonlinear oscillator typified by a mass attached to a stretched wire

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Abstract

The first-order harmonic balance method via the first Fourier coefficient is used to construct two approximate frequency–amplitude relations for a conservative nonlinear oscillatory system in which the restoring force has an irrational form. This system corresponds to the motion of a mass attached to a stretched wire. Two procedures are used to approximately solve the nonlinear differential equation. In the first, the differential equation is rewritten in a form that does not contain the square-root expression, while in the second the differential equation is solved directly. The approximate frequency obtained using the second procedure is more accurate than the frequency obtained with the first due to the fact that, in the second procedure, application of the harmonic balance method produces an infinite set of harmonics, while in the first procedure only two harmonics are produced. Both approximate frequencies are valid for the complete range of oscillation amplitudes, and excellent agreement of the approximate frequencies with the exact one are demonstrated and discussed. The discrepancy between the second approximate frequency and the exact one never exceeds 2.2%.

1. Introduction

Consider the motion of a particle of mass m attached to the centre of a stretched elastic wire [1] of coefficient of stiffness equal to k. The length of the elastic wire when any force is applied to it is $2a$. We assume that the movement of the particle is one-dimensional and this is constrained to move only in the horizontal x direction. As we can see in Fig. 1, the ends of the wire are fixed a distance $2d$ a part. Length d can be major or equal to a. If $d = a$ the wire is not stretched for $x = 0$ and there is no tension in each part of it. However, if $d > a$ the wire is stretched for $x = 0$ and the tension in each part of the wire is $k(d–a)$. The equation of motion is given by the following nonlinear differential equation [1,2]:

$$m\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + 2kx - \frac{2kax}{\sqrt{d^2 + x^2}} = 0 \tag{1}$$

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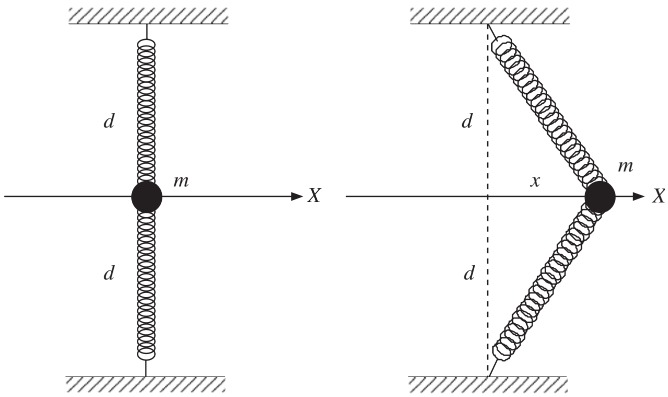

Fig. 1. Mass attached to a stretched wire.

with initial conditions

$$x(0) = x_0 \quad \text{and} \quad \frac{dx}{dt}(0) = 0. \tag{2}$$

Two dimensionless variables y and τ can be constructed as follows:

$$y = dx \quad \text{and} \quad \tau = \sqrt{\frac{2k}{m}}. \tag{3}$$

Substituting these dimensionless variables into Eq. (1) gives

$$\frac{d^2y}{d\tau^2} + y - \frac{\lambda y}{\sqrt{1+y^2}} = 0, \quad 0 < \lambda \leqslant 1 \tag{4}$$

with initial conditions

$$y(0) = A \quad \text{and} \quad \frac{dy}{d\tau}(0) = 0. \tag{5}$$

In Eqs. (4) and (5) we have defined the following parameters:

$$A = \frac{x_0}{d} \quad \text{and} \quad \lambda = \frac{a}{d}. \tag{6}$$

As $0 \leqslant a \leqslant d$ it follows that $0 < \lambda \leqslant 1$.

Eq. (4) is an example of a conservative nonlinear oscillatory system in which the restoring force has an irrational form [1,2] and this system and has the first integral

$$\frac{1}{2}\left(\frac{dy}{d\tau}\right)^2 + V(y) = E \geqslant 0, \tag{7}$$

where E is the "total energy" of the nonlinear oscillator and the potential function has the irrational form [1]

$$V(y) = \tfrac{1}{2}y^2 - \lambda\sqrt{1+y^2} + \lambda. \tag{8}$$

All the motions corresponding to Eq. (4) are periodic [1]; the system will oscillate within symmetric bounds $[-A, A]$, and the angular frequency and corresponding periodic solution of the nonlinear oscillator are dependent on the amplitude A.

For large x, and for $0 < \lambda \leqslant 1$, Eq. (4) approximates that of a linear harmonic oscillator

$$\frac{d^2y}{d\tau^2} + y = 0 \quad \text{for } y \gg 1 \text{ and } 0 < \lambda \leqslant 1, \tag{9}$$

so, for large A, we have $\omega \approx 1$. For small x, and for $0<\lambda<1$, the equation of motion also approximates that of a linear oscillator

$$\frac{d^2y}{d\tau^2}+(1-\lambda)y=0 \quad \text{for } y \ll 1 \text{ and } 0<\lambda<1 \tag{10}$$

and $\omega \approx \sqrt{1-\lambda}$ for small A. However, for small y, and for $\lambda=1$, Eq. (4) approximates that of a truly nonlinear oscillator

$$\frac{d^2y}{d\tau^2}+\frac{1}{2}y^3=0 \quad \text{for } x \ll 1 \text{ and } \lambda=1 \tag{11}$$

and ω tends to zero when A decreases. Consequently, the angular frequency ω increases from $\sqrt{1-\lambda}$ to 1 as the initial value of $y(0)=A$ increases.

It is difficult to solve nonlinear differential equations and, in general, it is often more difficult to get an analytic approximation than a numerical one for a given nonlinear oscillatory system [1,3]. There are many approaches for approximating solutions to nonlinear oscillatory systems. The most widely studied approximation methods are the perturbation methods [4]. The simplest and perhaps one of the most useful of these approximation methods is the Lindstedt–Poincaré perturbation method, whereby the solution is analytically expanded in the power series of a small parameter. To overcome this limitation, many new perturbative techniques have been developed. Modified Lindstedt–Poincaré techniques [5], homotopy perturbation method [6] or linear delta expansion [7–9] are only some examples of them. A recent detailed review of perturbation methods can be found in Ref. [10].

The harmonic balance method is another procedure for determining analytical approximations to the periodic solutions of differential equations by using a truncated Fourier series representation [1,11–17]. This method can be applied to nonlinear oscillatory systems where the nonlinear terms are not small and no perturbation parameter is required.

Since the restoring force is an odd function of y, the periodic solution $y(\tau)$ has the following Fourier series representation [15]:

$$y(\tau)=\sum_{n=0}^{\infty} h_{2n+1}\cos[(2n+1)\omega\tau] \tag{12}$$

which contains only odd multiples of $\omega\tau$. The purpose of the harmonic balance method is to approximate the periodic solution of Eq. (4) by a trigonometric polynomial

$$y(\tau)\approx\sum_{n=0}^{N} b_{2n+1}\cos[(2n+1)\omega\tau] \tag{13}$$

and determine both the coefficients b_{2n+1} and the angular frequency as a function of A.

The main objective of this paper is to solve Eq. (4) by applying the first-order harmonic balance method, and to compare the approximate frequency obtained with the exact one and with another approximate frequency obtained applying the harmonic balance method to the same oscillatory system but rewriting Eq. (4) in a way suggested previously by Mickens [1]. The approximate frequency derived here is more accurate and closer to the exact solution. The errors in the resulting frequency are reduced and the maximum relative error is less than 2.2% for all values of A and for the limiting case $\lambda=1$. We will see that the errors decrease when λ decreases and are as low as 0.58% for $\lambda=0.9$ or 0.062% for $\lambda=0.5$, for the complete range of oscillation amplitudes, including the limiting cases of amplitude approaching zero and infinity.

2. Solution method

Eq. (4) can be rewritten in a form that does not contain the square-root expression [1]

$$(1+y^2)\left(\frac{d^2y}{d\tau^2}+y\right)^2=\lambda^2y^2. \tag{14}$$

It is possible to solve Eq. (14) by applying the harmonic balance method. Following the lowest-order harmonic balance method, a reasonable and simple initial approximation satisfying the conditions in Eq. (5) would be

$$y(\tau) = A\cos\omega\tau. \tag{15}$$

The angular frequency of the oscillator is ω, which is unknown and to be determined. The corresponding period of the nonlinear oscillation is given by $T = 2\pi/\omega$. Both the periodic solution $y(\tau)$ and frequency ω (thus period T) depends on A. Substituting Eq. (15) into Eq. (14), then expanding and simplifying the resulting expression gives

$$\left[(-\omega^2+1)^2\left(1+\frac{3A^2}{4}\right)-\lambda^2\right] + (\text{higher-order harmonics}) = 0. \tag{16}$$

From Eqs. (9)–(11) and (16), the solution for ω is

$$\omega_1(A) = \sqrt{1-\lambda\left(1+\tfrac{3}{4}A^2\right)^{-1/2}} \tag{17}$$

which is valid for the whole range of values of λ ($0<\lambda\leqslant 1$). The approximate frequency in Eq. (17) is more accurate than the approximate frequency obtained in Ref. [1].

As we pointed out in the introduction, the main objective of this paper is to solve Eq. (4) instead of Eq. (14) by applying the harmonic balance method. Substitution of Eq. (15) into Eq. (4) gives

$$-A\omega^2\cos\omega\tau + A\cos\omega\tau - \frac{\lambda A\cos\omega\tau}{\sqrt{1+\cos^2\omega\tau}} = 0. \tag{18}$$

The power-series expansion of $y/\sqrt{1+y^2}$ is

$$\frac{y}{\sqrt{1+y^2}} = y + \sum_{n=1}^{\infty}(-1)^n\frac{(2n-1)!}{2^{2n-1}n!(n-1)!}y^{2n+1}. \tag{19}$$

Substituting Eq. (19) into Eq. (18) and taking into account Eq. (15) gives

$$-\omega^2\cos\omega\tau + \cos\omega\tau - \lambda\cos\omega\tau - \lambda\sum_{n=1}^{\infty}(-1)^n\frac{(2n-1)!}{2^{2n-1}n!(n-1)!}A^{2n}\cos^{2n+1}\omega\tau = 0. \tag{20}$$

The formula that allows us to obtain the odd power of the cosine is

$$\cos^{2n+1}\omega\tau = \frac{1}{2^{2n}}\left\{\binom{2n+1}{n}\cos\omega\tau + \binom{2n+1}{n-1}\cos 3\omega\tau + \cdots + \binom{2n+1}{0}\cos[(2n+1)\omega\tau]\right\}. \tag{21}$$

Substituting Eq. (21) into Eq. (20) gives

$$\left[-\omega^2 + 1 - \lambda\sum_{n=0}^{\infty}c_{2n+1}A^{2n}\right]\cos\omega\tau + (\text{higher-order harmonics}) = 0, \tag{22}$$

where the coefficients c_{2n+1} are given by

$$c_1 = 1 \tag{23}$$

and

$$c_{2n+1} = (-1)^n\frac{(2n-1)!(2n+1)!}{2^{4n-1}(n!)^2(n-1)!(n+1)!} \quad \text{for } n\geqslant 1. \tag{24}$$

For the lowest-order harmonic to be equal to zero, it is necessary to set the coefficient of $\cos\omega\tau$ equal to zero in Eq. (22), then

$$\omega = \left(1-\lambda\sum_{n=0}^{\infty}c_{2n+1}A^{2n}\right)^{1/2}. \tag{25}$$

In order to obtain the value of $\sum_{n=0}^{\infty} c_{2n+1}A^{2n}$ in Eq. (25) we consider the following relations:

$$\frac{(2n-1)!}{2^{2n-1}(n-1)!} = (1/2)_n, \quad \frac{(2n+1)!}{2^{2n}n!} = (3/2)_n, \quad (n+1)! = (2)_n, \tag{26}$$

where $(a)_n$ is the Pochhammer symbol [18]

$$(a)_n = a(a+1)\cdots(a+n-1). \tag{27}$$

Taking into account Eqs. (23), (26) and (27), it is possible to write $\sum_{n=0}^{\infty} c_{2n+1}A^{2n}$ as follows:

$$\sum_{n=0}^{\infty} c_{2n+1}A^{2n} = \sum_{n=0}^{\infty} \frac{(1/2)_n(3/2)_n}{(2)_n}\frac{(-A^2)^n}{n!} = {}_2F_1\left(\frac{1}{2},\frac{3}{2};2;-A^2\right), \tag{28}$$

where ${}_2F_1(a,b;c;z)$ is the hypergeometric function [18]

$${}_2F_1(a,b;c;z) = \sum_{n=1}^{\infty} \frac{(a)_n(b)_n}{(c)_n}\frac{z^n}{n!}. \tag{29}$$

Substituting Eq. (28) into Eq. (25) gives

$$\omega_2(A) = \left[1 - \lambda_2 F_1\left(\tfrac{1}{2},\tfrac{3}{2};2;-A^2\right)\right]^{1/2}, \tag{30}$$

which is the angular frequency obtained applying the first-order harmonic balance method directly to Eq. (4).

3. Results and discussion

In this section, we illustrate the accuracy of the proposed approach by comparing the approximate frequencies $\omega_1(A)$ and $\omega_2(A)$ obtained in this paper with the exact frequency $\omega_e(A)$. The exact angular frequency is calculated as follows. Integrating Eq. (4) and using the initial conditions in Eq. (5), we arrive at

$$\frac{1}{2}\left(\frac{dy}{d\tau}\right)^2 + \frac{1}{2}y^2 - \lambda\sqrt{1+y^2} = \frac{1}{2}A^2 - \lambda\sqrt{1+A^2}. \tag{31}$$

The exact frequency can then be derived as follows:

$$\omega_e(A) = \frac{\pi}{2}\left[\int_0^1 \frac{A\,\mathrm{d}u}{\sqrt{A^2(1-u^2) - 2\lambda(\sqrt{1+A^2} - \sqrt{1+A^2u^2})}}\right]^{-1}. \tag{32}$$

For small values of the amplitude A it is possible to take into account the following approximation, which is valid for $0<\lambda<1$:

$$\omega_e(A) \approx \frac{\pi}{2}\left[\int_0^1 \frac{\mathrm{d}u}{\sqrt{1-u^2}}\left(\frac{1}{\sqrt{1-\lambda}} - \frac{\lambda(1+u^2)}{8(1-\lambda)^{3/2}}A^2 - \frac{\lambda[2(\lambda-4)u^2 + (5\lambda-8)(1+u^4)]}{128(1-\lambda)^{5/2}}A^4\cdots\right)\right]^{-1} \tag{33}$$

and the following frequency for $\lambda = 1$

$$\omega_e(A) \approx \frac{\pi}{2}\left[\int_0^1 \frac{2\,\mathrm{d}u}{\sqrt{(1-u^2)(1+u^2)}}\left(\frac{1}{A}+\cdots\right)\right]^{-1}. \tag{34}$$

The power-series expansions of the exact angular frequency, ω_e, are

$$\omega_e(A) \approx \sqrt{1-\lambda} + \frac{3\lambda}{16\sqrt{1-\lambda}}A^2 + \frac{3\lambda(33\lambda-40)}{1024(1-\lambda)^{3/2}}A^4 + \cdots \quad \text{for } 0<\lambda<1 \tag{35}$$

and

$$\omega_e(A) \approx \frac{\pi}{4K(-1)}A + \cdots = 0.59907A + \cdots \quad \text{for } \lambda = 1, \tag{36}$$

where $K(m)$ is the complete elliptical integral of the first kind [19]

$$K(m)=\int_0^1 \frac{\mathrm{d}z}{\sqrt{(1-z^2)(1-mz^2)}}. \tag{37}$$

For small values of A it is also possible to do the power-series expansion of the approximate angular frequencies ω_1 (Eq. (17)) and ω_2 (Eq. (30)). In this way, the following equations can be obtained:

$$\omega_1(A)\approx\sqrt{1-\lambda}+\frac{3\lambda}{16\sqrt{1-\lambda}}A^2+\frac{3\lambda(30\lambda-36)}{1024(1-\lambda)^{3/2}}A^4+\cdots \quad \text{for } 0<\lambda<1, \tag{38}$$

$$\omega_2(A)\approx\sqrt{1-\lambda}+\frac{3\lambda}{16\sqrt{1-\lambda}}A^2+\frac{3\lambda(34\lambda-40)}{1024(1-\lambda)^{3/2}}A^4+\cdots \quad \text{for } 0<\lambda<1 \tag{39}$$

and

$$\omega_1(A)\approx\sqrt{\tfrac{3}{8}}A+\cdots=0.61237A+\cdots \quad \text{for } \lambda=1, \tag{40}$$

$$\omega_2(A)\approx\sqrt{\tfrac{3}{8}}A+\cdots=0.61237A+\cdots \quad \text{for } \lambda=1. \tag{41}$$

As can be seen, in the expansions of the angular frequencies for $0<\lambda<1$, ω_1 (Eq. (38)) and ω_2 (Eq. (39)), the first two terms are the same as the first two terms of the equation obtained in the power-series expansion of the exact angular frequency, ω_e (Eq. (35)). If we compare the third terms in Eqs. (38) and (39) with the third term in the series expansion of the exact frequency ω_e (Eq. (35)), we can see that the third term in the series expansions of ω_2 (Eq. (39)) is more accurate than the third term in the expansion of ω_1 (Eq. (38)). On the other hand, if we compare the angular frequencies for $\lambda=1$ (Eqs. (36), (40) and (41)), we can see that the relative error is 2.2% for $A\to 0$.

Now we are going to obtain an asymptotic representation for large amplitudes. We consider the expression for the exact frequency ω_e (Eq. (32)) and we do the change $A=1/B$. For large amplitudes ($A\to\infty$) we have $B\to 0$. Taking this into account, and doing the power-series expansion of the result for small values of B, we obtain

$$\begin{aligned}\omega_e(A)&=\frac{\pi}{2}\left[\int_0^1 \frac{A\,\mathrm{d}u}{\sqrt{A^2(1-u^2)-2\lambda(\sqrt{1+A^2}-\sqrt{1+A^2u^2})}}\right]^{-1}\\&=\frac{\pi}{2}\left[\int_0^1 \frac{1}{\sqrt{1-u^2-2\lambda B(\sqrt{1+B^2}-\sqrt{B^2+u^2})}}\right]^{-1}\\&\approx\frac{\pi}{2}\left[\int_0^1\left(\frac{1}{\sqrt{1-u^2}}-\frac{\lambda B}{(1+u)\sqrt{1-u^2}}+\frac{3\lambda^2B^2}{2(1+u)^2\sqrt{1-u^2}}+\cdots\right)\mathrm{d}u\right]^{-1}.\end{aligned} \tag{42}$$

The power-series expansion for the exact frequency for small values of B (large values of A) is

$$\begin{aligned}\omega_e(A)&\approx\frac{\pi}{2}\left(\frac{\pi}{2}+\lambda B+\lambda^2B^2+\cdots\right)^{-1}\approx 1-\frac{2\lambda}{\pi A}-\frac{2(\pi-2)\lambda^2}{\pi^2A^2}+\cdots\\&=1-\frac{0.63662\lambda}{A}-\frac{0.23134\lambda^2}{A^2}+\cdots.\end{aligned} \tag{43}$$

Substituting $A=1/B$ in ω_1 (Eq. (17)) and ω_2 (Eq. (30)) and doing the power-series asymptotic expansions for small values of B (large values of A) using the MATHEMATICA program, it is easy to obtain the following approximations valid for large amplitudes:

$$\omega_1(A) \approx 1 - \frac{\lambda}{\sqrt{3}A} - \frac{\lambda^2}{6A^2} + \cdots = 1 - \frac{0.57735\,\lambda}{A} - \frac{0.16667\,\lambda^2}{A^2} + \cdots, \tag{44}$$

$$\omega_2(A) \approx 1 - \frac{2\lambda}{\pi A} - \frac{2\lambda^2}{\pi^2 A^2} + \cdots = 1 - \frac{0.63662\,\lambda}{A} - \frac{0.20264\,\lambda^2}{A^2} + \cdots. \tag{45}$$

As we can see, in the expansion of the angular frequency ω_1 for large amplitudes (Eq. (44)), only the first term is the same as the first term of the equation obtained in the power-series expansion of the exact frequency, ω_e (Eq. (43)). However in the expansion of the angular frequency ω_2 (Eq. (45)), the first two terms are the same as the first two terms of the equation obtained in the power-series expansion of the exact frequency, ω_e (Eq. (44)). These results confirm the fact that ω_2 is a better approximation to the exact frequency ω_e than the approximate frequency ω_1, not only for small amplitudes but also for large values of the amplitude of oscillation.

From Eqs. (43)–(45) and for $0<\lambda\leqslant 1$, it is easy to obtain the following expressions:

$$\lim_{A\to\infty} \omega_e(A) = \lim_{A\to\infty} \omega_1(A) = \lim_{A\to\infty} \omega_2(A) = 1 \tag{46}$$

and

$$\lim_{A\to\infty} \frac{\omega_1}{\omega_e} = \lim_{A\to\infty} \frac{\omega_2}{\omega_e} = 1. \tag{47}$$

From Eqs. (35), (38) and (39) it is easy to see that

$$\lim_{A\to 0} \omega_e(A) = \lim_{A\to 0} \omega_1(A) = \lim_{A\to 0} \omega_2(A) = \sqrt{1-\lambda} \tag{48}$$

and

$$\lim_{A\to 0} \frac{\omega_1}{\omega_e} = \lim_{A\to 0} \frac{\omega_2}{\omega_e} = 1 \tag{49}$$

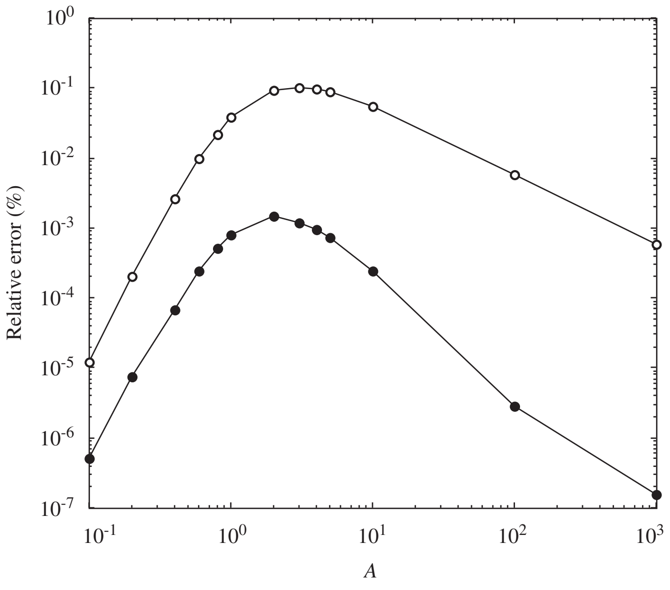

Fig. 2. Relative error for approximate frequencies ω_1 (○) and ω_2 (●) and for $\lambda = 0$.

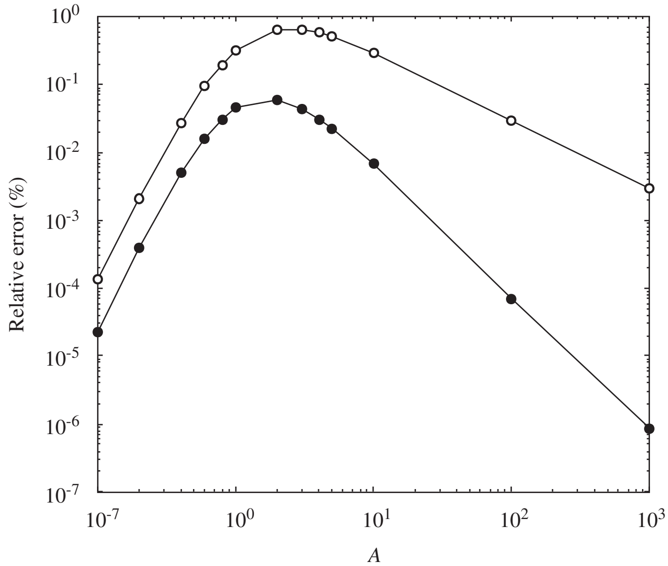

Fig. 3. Relative error for approximate frequencies ω_1 (○) and ω_2 (●) and for $\lambda = 0.5$.

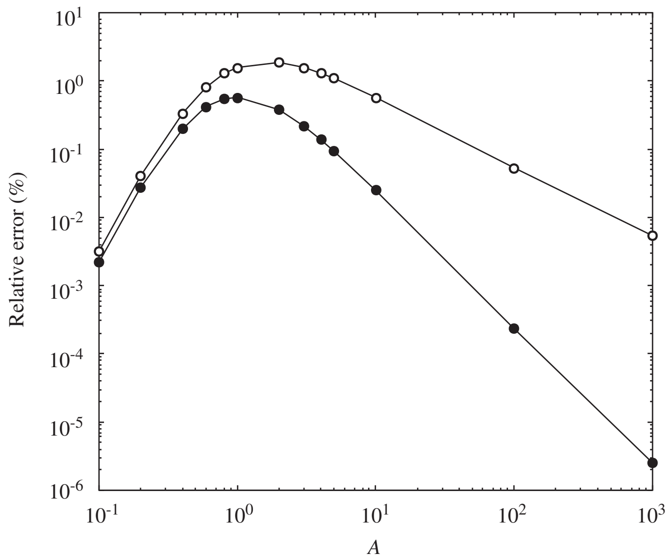

Fig. 4. Relative error for approximate frequencies ω_1 (○) and ω_2 (●) and for $\lambda = 0.9$.

for $0 < \lambda < 1$, while from Eqs. (36), (40) and (41), the following relation is satisfied:

$$\lim_{A \to 0} \frac{\omega_1}{\omega_e} = \lim_{A \to 0} \frac{\omega_2}{\omega_e} = 1.0222 \tag{50}$$

for $\lambda = 1$.

In Figs. 2–5 we have plotted the percentage error of approximate frequencies ω_1 and ω_2, calculated using Eqs. (17) and (30), respectively, for $0.1 \leqslant A \leqslant 1000$ and four different values of the parameter $\lambda = 0.1$, 0.5, 0.9 and 1. In these figures, the percentage errors were calculated using the following equation:

$$\text{Relative error of } \omega_j\ (\%) = 100\left|\frac{\omega_j - \omega_e}{\omega_e}\right|, \quad j = 1, 2. \tag{51}$$

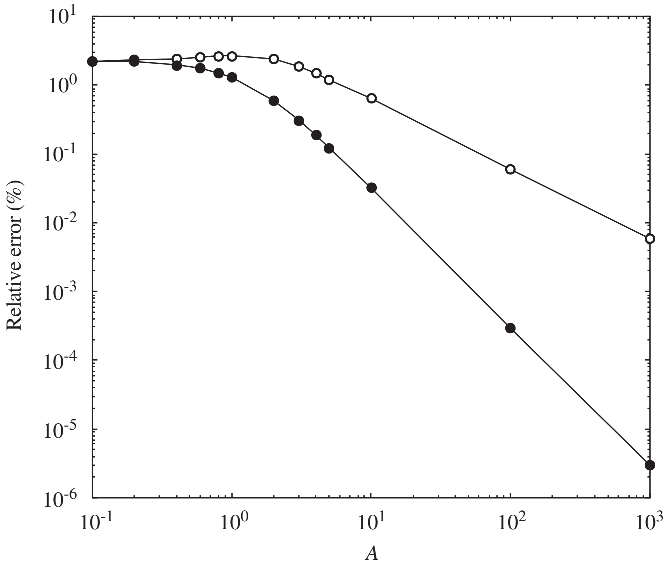

Fig. 5. Relative error for approximate frequencies ω_1 (○) and ω_2 (●) and for $\lambda = 1$.

Table 1
Maximum relative error in the approximate frequencies ω_1 and ω_2 for different values of λ

λ	Maximum relative error of ω_1 (%)	A_1	Maximum relative error of ω_2 (%)	A_2
0.01	0.0099	2.98	0.000013	2.06
0.1	0.10	2.89	0.0014	2.03
0.2	0.22	2.79	0.0065	1.96
0.3	0.35	2.69	0.017	1.86
0.4	0.50	2.57	0.034	1.77
0.5	0.67	2.47	0.062	1.67
0.6	0.87	2.30	0.11	1.55
0.7	1.1	2.14	0.18	1.42
0.8	1.4	1.93	0.31	1.26
0.9	1.9	1.67	0.58	1.02
0.99	2.6	1.20	1.5	0.54
0.999	2.7	1.09	1.9	0.31
0.9999	2.7	1.07	2.1	0.17
1	2.7	1.07	2.2	0.00

A_1 and A_2 are the values of A for which the relative error in ω_1 and ω_2, respectively, is maximum.

Table 1 includes the maximum relative errors for ω_1 and ω_2 and for values of λ between 0.1 and 1. To obtain the values included in Table 1 we plotted the relative error as a function of A and we chose the maximum value of the relative error for each λ. As we can see in Table 1, for a fixed value of A the relative error increases when λ increases. Figs. 2–5 and Table 1 indicate that Eq. (30) is more accurate than Eq. (17) and can provide excellent approximations to the exact frequency for small as well as large oscillation amplitudes. From Table 1 we can conclude that the relative errors for the approximate frequency ω_1 are lower than 2.7% for $0<\lambda\leqslant 1$; while for ω_2 these errors are lower than 2.2% for the same range of values of λ. However, we can see that the approximate frequency ω_1 is less accurate than ω_2. For example, for $\lambda = 0.9$ the maximum relative error in ω_2 is as low as 0.58%, while the relative error in ω_1 is 1.9%. Figs. 2–5 and Table 1 show that ω_2 gives excellent approximate frequencies for small as well as large values of oscillation amplitude A and for the whole range of values of λ.

At this point it is necessary to answer the following questions: (a) why does substitution of Eq. (15) into Eq. (14) not give the same result as substitution of Eq. (15) into Eq. (4)?, and (b) why does application of the first-order harmonic balance method to Eq. (4) give a more accurate frequency than application of the method to Eq. (14)? To answer these questions we substitute Eq. (15) into Eq. (14) again. This substitution gives

$$(1 + A^2\cos^2\omega\tau)(-\omega^2 + 1)^2\cos^2\omega\tau = \lambda^2\cos^2\omega\tau \tag{52}$$

If we divide this equation by $\cos\omega\tau$ we obtain

$$(1 + A^2\cos^2\omega\tau)(-\omega^2 + 1)^2 \cos\omega\tau = \lambda^2 \cos\omega\tau, \tag{53}$$

which can be written as follows:

$$[(-\omega^2 + 1)^2 - \lambda^2] \cos\omega\tau + A^2(-\omega^2 + 1)^2 \cos^3\omega\tau = 0. \tag{54}$$

This equation includes only two odd powers of $\cos\omega\tau$, $\cos\omega\tau$ and $\cos^3\omega\tau$, and then there are only two contributions to the coefficient of the first harmonic $\cos\omega\tau$, which are 1 from $\cos\omega\tau$ and 3/4 from $\cos^3\omega\tau$. Therefore, substituting Eq. (15) into Eq. (14) produces only the first harmonic, $\cos\omega\tau$, and the third harmonic, $\cos 3\omega\tau$,

$$\left[(-\omega^2 + 1)^2\left(1 + \tfrac{3}{4}A^2\right) - \lambda^2\right] \cos\omega\tau + \tfrac{1}{4}(-\omega^2 + 1)^2A^2 \cos 3\omega\tau = 0. \tag{55}$$

Setting the coefficient of $\cos\omega\tau$ equal to zero gives

$$(-\omega^2 + 1)\left(1 + \tfrac{3}{4}A^2\right)^{1/2} - \lambda = 0, \tag{56}$$

which gives the approximate frequency ω_1 in Eq. (17).

Now we consider Eq. (20) again. This equation can be written as follows:

$$(-\omega^2 + 1 - \lambda) \cos\omega\tau - \lambda\sum_{n=1}^{\infty} c_{2n+1}A^{2n}\cos^{2n+1}\omega\tau = 0. \tag{57}$$

This equation includes all odd powers of $\cos\omega\tau$, which are $\cos^{2n+1}\omega\tau$ with $n = 0, 1, 2, \ldots, \infty$, and then there are infinite contributions to the coefficient of the first harmonic $\cos\omega t$, that is, 1 from $\cos\omega\tau$, 3/4 from $\cos^3\omega\tau$, 5/8 from $\cos^5\omega\tau, \ldots, 2^{-2n}\begin{pmatrix} 2n+1 \\ n \end{pmatrix}$ from $\cos^{2n+1}\omega\tau$, and so on. Therefore, substituting Eq. (15) into Eq. (4) produces the infinite set of higher harmonics, $\cos\omega\tau, \cos 3\omega\tau, \ldots, \cos[(2n+1)\omega\tau]$, and so on. Similar phenomenon occurred in Ref. [16] for the Duffing-harmonic oscillator.

Substituting Eq. (21) into Eq. (20) and taking into account Eq. (28) gives

$$[-\omega^2 + 1 - \lambda a_1(A)] \cos\omega\tau - \lambda\sum_{n=1}^{\infty} a_{2n+1}(A) \cos[(2n+1)\omega\tau] = 0, \tag{58}$$

where

$$a_1(A) = \sum_{n=0}^{\infty} c_{2n+1}A^{2n} = {}_2F_1\left(\tfrac{1}{2}, \tfrac{3}{2}; 2; -A^2\right) \tag{59}$$

and the coefficients a_{2n+1} for $n \geqslant 1$ can be obtained taking into account Eqs. (24) and (25). Setting the coefficient of $\cos\omega\tau$ equal to zero gives

$$-\omega^2 + 1 - \lambda\,{}_2F_1\left(\tfrac{1}{2}, \tfrac{3}{2}; 2; -A^2\right) = 0. \tag{60}$$

This equation can be rewritten as follows:

$$(-\omega^2 + 1)\left[\left({}_2F_1\left(\tfrac{1}{2}, \tfrac{3}{2}; 2; -A^2\right)\right)^{-2}\right]^{1/2} - \lambda = 0. \tag{61}$$

It can be seen that Eqs. (56) and (61) have the form

$$(-\omega^2 + 1)(f(A))^{1/2} - \lambda = 0, \tag{62}$$

which allows the approximate frequency ω to be determined in terms of the oscillation amplitude A

$$\omega(A) = \sqrt{1 - \lambda(f(A))^{-1/2}}. \tag{63}$$

From this equation we can conclude that application of the first-order harmonic balance method to Eqs. (4) and (14) gives the same functional form for the approximate frequency ω. The difference between these approximate frequencies is the function $f(A)$

$$\omega_1(A) = \sqrt{1 - \lambda\left(1 + \tfrac{3}{4}A^2\right)^{-1/2}} \quad \text{and} \quad f(A) = 1 + \tfrac{3}{4}A^2, \tag{64}$$

$$\omega_2(A) = \sqrt{1 - \lambda\left({}_2F_1\left(\tfrac{1}{2}, \tfrac{3}{2}; 2; -A^2\right)^{-2}\right)^{-1/2}} \quad \text{and} \quad f(A) = {}_2F_1\left(\tfrac{1}{2}, \tfrac{3}{2}; 2; -A^2\right)^{-2}. \tag{65}$$

We can do the following power-series expansion:

$${}_2F_1\left(\tfrac{1}{2}, \tfrac{3}{2}; 2; -A^2\right)^{-2} = 1 + \tfrac{3}{4}A^2 - \tfrac{3}{64}A^4 + \tfrac{13}{512}A^6 + \cdots. \tag{66}$$

Substituting Eq. (66) into Eq. (61), we have

$$(-\omega^2 + 1)\left(1 + \tfrac{3}{4}A^2 - \tfrac{3}{64}A^4 + \tfrac{13}{512}A^6 + \cdots\right)^{1/2} - \lambda = 0. \tag{67}$$

As can be seen, in this equation the first two terms in brackets are identical to the two terms in brackets in Eq. (56); whereas powers $A^4, A^6, \ldots$ are due to the infinite set of higher harmonics in Eq. (20). Applying the harmonic balance method to Eqs. (4) and (14) with higher harmonics, the two procedures will give more accurate results [16]. In the limit in which we include all the harmonics, they must give us exactly the same solution, since Eq. (14) is equivalent to Eq. (4).

4. Conclusions

The first-order harmonic balance method was used to obtain two approximate frequencies for a conservative nonlinear oscillatory system in which the restoring force has an irrational form. The first approximate frequency, ω_1, was obtained by rewriting the nonlinear differential equation in a form that does not contain an irrational expression; while the second one, ω_2, was obtained by solving the nonlinear differential equation containing a square-root expression approximately. We can conclude that formulas (17) and (30) are valid for the complete range of oscillation amplitude, including the limiting cases of amplitude approaching zero and infinity. Excellent agreement of the approximate frequencies with the exact one was demonstrated and discussed and the discrepancy between the second approximate frequency, ω_2, and the exact one never exceeds 2.2%. For example, the maximum relative error for this frequency is as low as 0.062% for $\lambda = 0.5$ and all values of the amplitude A; while for the first approximate frequency, ω_1, this maximum relative error is 0.67%, ten times more. The second approximate frequency, ω_2, derived here is the best frequency that can be obtained using the first-order harmonic balance method, and the maximum relative error was significantly reduced as compared with the first approximate frequency, ω_1. Finally, we discussed the reason why the accuracy of the second approximate frequency, ω_2, is better than that of the first frequency, ω_1. This reason is related to the number of harmonics that application of the first-order harmonic balance method produces for each differential equation solved, two harmonics for the first case and the infinite set of harmonics for the second one.

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